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GROUPS, SEMILATTICES AND INVERSE SEMIGROUPS

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ABSTRACT. An inverse semigroup S is called proper if the equations $ea = e = e^2$ together imply $a^2 = a$ for each $a, e \in S$. In this paper a construction is given for a large class of proper inverse semigroups in terms of groups and partially ordered sets; the semigroups in this class are called P-semigroups. It is shown that every inverse semigroup divides a P-semigroup in the sense that it is the image, under an idempotent separating homomorphism, of a full subsemigroup of a P-semigroup. Explicit divisions of this type are given for ω-bisimple semigroups, proper bisimple inverse semigroups, semilattices of groups and Brandt semigroups.

The algebraic theory of inverse semigroups has been the subject of a considerable amount of investigation in recent years. Much of this research has been concerned with the problem of describing inverse semigroups in terms of their semilattice of idempotents and maximal subgroups. For example, Clifford [2] described the structure of all inverse semigroups with central idempotents (semilattices of groups) in these terms. Reilly [17] described all ω-bisimple inverse semigroups in terms of their idempotents while Warne [22], [23] considered I-bisimple, ω^n-bisimple and ω^n-I-bisimple inverse semigroups; McAlister [8] described arbitrary 0-bisimple inverse semigroups in a similar way. Further, Munn [12] and Kochin [6] showed that ω-inverse semigroups could be explicitly described in terms of their idempotents and maximal subgroups. Their results have since been generalised by Ault and Petrich [1], Lallement [7] and Warne [24] to other restricted classes of inverse semigroups. Munn [14] has also described various classes of simple inverse semigroups in terms of groups and their semilattice of idempotents.

A homomorphism θ of an inverse semigroup S into an inverse semigroup T is called *idempotent separating* if θ is one-to-one on the idempotents of S; that is, $e^2 = e, f^2 = f, e\theta = f\theta$ imply $e = f$. A congruence is idempotent separating if it is the congruence of an idempotent separating homomorphism. Howie [5] showed that

$$\mu = \{(a,b) \in S \times S\colon a^{-1}ea = b^{-1}eb \text{ for all } e^2 = e \in S\}$$

is the maximum idempotent separating congruence on an inverse semigroup S. Further, Munn [13] has shown that $T = S/\mu$ is *fundamental*; that is, the identity is the only congruence on T contained in $\mathcal{H}$. The importance of fundamental inverse semigroups stems from the fact (Munn [11], [13]) that, if T is a

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fundamental inverse semigroup with idempotents E, then T is isomorphic to a full subsemigroup of the semigroup T_E of isomorphisms between principal ideals of E. (An inverse subsemigroup U of an inverse semigroup V is *full* if it contains all the idempotents of V.) Since the kernel of an idempotent separating homomorphism is a semilattice of groups, it follows that any inverse semigroup S with semilattice of idempotents E is an idempotent separating extension of a semilattice E of groups by a full subsemigroup of T_E.

Coudron [3] and D'Alarcao [4] have given extension theories similar to the Schreier theory for groups, for idempotent separating extensions of inverse semigroups. Thus, assuming that the full subsemigroups of T_E can be found, Coudron and D'Alarcao's extension theories can be regarded as giving the structure of all inverse semigroups.

In this paper, we present a structure theory for inverse semigroups which is dual to that of Munn-Coudron-D'Alarcao which is outlined above. Given a group G acting on a partially ordered set $\mathcal{X}$ by order automorphisms and a subsemilattice $\mathcal{Y}$ of $\mathcal{X}$, we construct an inverse semigroup $P(G, \mathcal{X}, \mathcal{Y})$ which depends only on $\mathcal{Y}$ and the action of G on $\mathcal{X}$. The main theorem (Theorem 2.4) shows that any inverse semigroup S, with semilattice of idempotents E, is an idempotent separating homomorphic image of a full subsemigroup of $P(G, \mathcal{X}, E)$ for some G, $\mathcal{X}$. Since idempotent separating congruences on inverse semigroups are known, thanks to Preston [16], our theory can be regarded as giving the structure of all inverse semigroups (provided that we assume that full subsemigroups of the semigroups $P(G, \mathcal{X}, \mathcal{Y})$ are known). This structure theory can be regarded as an extension, to infinite inverse semigroups, and refinement of Rhodes divisibility theory for finite semigroups [19].

H. E. Scheiblich [20] has recently given an explicit description of the free inverse semigroup I_X on a nonempty set X. The semigroups $P(G, \mathcal{X}, \mathcal{Y})$ which we consider in this paper generalise the semigroup used in his description of I_X. The results described here point out the fundamental nature of the construction, which he pioneered in his determination of I_X, to the structure theory of inverse semigroups in general.

In §§3, 4, 5 we explicitly construct divisions, of the type outlined above, for semilattices of groups, bisimple inverse semigroups and Brandt semigroups. The theory also gives rise to many interesting questions concerning partially ordered sets as well as inverse semigroups. Some of these are listed at the end of the paper.

1. The semigroups $P(G, \mathcal{X}, \mathcal{Y})$. Let $\mathcal{X}$ be a partially ordered set. For $A, B \in \mathcal{X}$, we shall denote by $A \wedge B$ the meet of A and B if it exists. If $\mathcal{Y}$ is a subset of $\mathcal{X}$, the statement "$A \wedge B \in \mathcal{Y}$" means that the meet of A and B exists and belongs to $\mathcal{Y}$. A subset $\mathcal{Y}$ of $\mathcal{X}$ is called a subsemilattice of $\mathcal{X}$ if $A, B \in \mathcal{Y}$ implies $A \wedge B \in \mathcal{Y}$ for all $A, B \in \mathcal{Y}$; $\mathcal{Y}$ is an ideal of $\mathcal{X}$ if $A \in \mathcal{Y}$ and $B \leq A$ together imply $B \in \mathcal{Y}$.

Let $\mathcal{X}$ be a partially ordered set and let G be a group which acts (on the left) on $\mathcal{X}$ by order automorphisms; let $\mathcal{Y}$ be a subsemilattice of $\mathcal{X}$. Then

$$\begin{aligned} P &= P(G,\mathcal{X},\mathcal{Y}) \\ &= \{(A,g) \in \mathcal{Y}\times G\colon A \wedge gB, g^{-1}(A \wedge B) \in \mathcal{Y} \text{ for all } B \in \mathcal{Y}\} \end{aligned}$$

under the multiplication $(A,g)(B,h) = (A \wedge gB, gh)$. Lemma 1.1 shows that P is an inverse semigroup; we shall call it a P-semigroup. When $\mathcal{Y}$ is not only a subsemilattice but an ideal of $\mathcal{X}$ the definition of P simplifies considerably; in this case $P = \{(A,g) \in \mathcal{Y}\times G\colon A, g^{-1}A \in \mathcal{Y}\}$. Thus P-semigroups are generalisations of the inverse semigroups constructed in [9, §4]. In particular, free inverse semigroups are P-semigroups [20], [9].

Lemma 1.1. *$P = P(G,\mathcal{X},\mathcal{Y})$ is an inverse semigroup with semilattice of idempotents isomorphic to $\mathcal{Y}$. For each $(A,g) \in P$, $(A,g)^{-1} = (g^{-1}A, g^{-1})$.*

Proof. Suppose (A,g), $(B,h) \in P$ and let $C \in \mathcal{Y}$. Then

$$\begin{aligned} (A \wedge gB) \wedge ghC &= A \wedge g(B \wedge hC) \\ &= A \wedge gD \quad \text{where } D = B \wedge hC \in \mathcal{Y} \text{ since } (B,h) \in P \\ &\in \mathcal{Y} \quad \text{since } (A,g) \in P \text{ and } D \in \mathcal{Y}. \end{aligned}$$

Further

$$\begin{aligned} (gh)^{-1}[(A \wedge gB) \wedge C] &= h^{-1}[B \wedge g^{-1}(A \wedge C)] \\ &= h^{-1}[B \wedge F] \quad \text{where } F = g^{-1}(A \wedge C) \in \mathcal{Y} \\ &\qquad\qquad \text{since } (A,g) \in P \\ &\in \mathcal{Y} \quad \text{since } (B,h) \in \mathcal{Y}, F \in \mathcal{Y}. \end{aligned}$$

Hence P is closed under multiplication.

Next

$$\begin{aligned} [(A,g)(B,h)](C,k) &= (A \wedge gB, gh)(c,k) = ((A \wedge gB) \wedge ghC, ghk) \\ &= (A \wedge g(B \wedge hC), ghk) = (A,g)[(B,h)(C,k)]. \end{aligned}$$

Hence the multiplication is associative.

Further $(A,g)^2 = (A \wedge gA, g^2) = (A,g)$ if and only if $g = 1$ and $(A,1)(B,1) = (A \wedge B, 1)$. Hence P has idempotents $\{(A,1)\colon A \in \mathcal{Y}\}$ and these form a semilattice isomorphic to $\mathcal{Y}$.

Let $(A,g) \in P$; then it is easy to see that $(g^{-1}A, g^{-1}) \in P$ and

$$(A,g)(g^{-1}A,g^{-1})(A,g) = (A,1)(A,g) = (A,g),$$
$$(g^{-1}A,g^{-1})(A,g)(g^{-1}A,g^{-1}) = (g^{-1}A,g^{-1})(A,1) = (g^{-1}A,g^{-1}).$$

Hence P is inverse and $(A,g)^{-1} = (g^{-1}A,g^{-1})$.

In the remainder of this section, we concentrate on describing the structure of full subsemigroups of P-semigroups.

Suppose that T is a full subsemigroup of $P(G, \mathcal{X}, \mathcal{Y})$ and let $A \in \mathcal{Y}$. Then we shall denote by G_A the set $\{g \in G: (A,g) \in T\}$; $H_A = \{g \in G: gA = A\}$ is the stabilizer of A.

Proposition 1.2. *Let T be a full inverse subsemigroup of $P(G, \mathcal{X}, \mathcal{Y})$. Then in T,*

(i) $(A,g)\ \mathcal{R}\ (B,h) \Leftrightarrow A = B$;

(ii) $(A,g)\ \mathcal{L}\ (B,h) \Leftrightarrow g^{-1}A = h^{-1}B$;

(iii) $(A,g)\ \mathcal{H}\ (A,1) \Leftrightarrow gA = A \Leftrightarrow g \in G_A \cap H_A$; *the maximal subgroup containing $(A, 1)$ is isomorphic to $G_A \cap H_A$;*

(iv) $(A,1)\ \mathcal{D}\ (B,1) \Leftrightarrow A = gB$ *for some* $g \in G_A$;

(v) $(A,1) \leq_{\mathcal{J}} (B,1) \Leftrightarrow A \leq gB$ *for some* $g \in G_A$.

We omit the straightforward proof; it is similar to that of [9, Proposition 2.6]. When $\mathcal{Y}$ is an ideal of $\mathcal{X}$ and $T = P(G,\mathcal{X},\mathcal{Y})$, we can omit the requirement that $g \in G_A$ in each of (iii), (iv), (v).

Munn [10] has shown that the relation $\sigma = \{(a,b) \in S \times S: ae = be$ for some $e^2 = e \in S\}$ is the finest group congruence on any inverse semigroup.

Proposition 1.3. *Let T be a full inverse subsemigroup of $P(G, \mathcal{X}, \mathcal{Y})$ and let $G' = \{g \in G: (A,g) \in T$ for some $A \in \mathcal{Y}\}$. Then G' is isomorphic to the maximal group homomorphic image of T.*

Proof. First of all, from the definition of multiplication in T, the map $\gamma: T \to G$ defined by $(A,g)\gamma = g$ is clearly a homomorphism with image G' so that G' is a group. Further, from the definition of σ,

$$(A,g)\sigma(B,h) \Leftrightarrow g = h \Leftrightarrow (A,g)\gamma = (B,h)\gamma.$$

Hence, by the fundamental homomorphism theorem for semigroups, $G' \approx T/\sigma$.

We shall say that an inverse subsemigroup T of an inverse semigroup S is a *large subsemigroup* of S if T is a full subsemigroup of S and each σ_S-class of S meets T. Thus, if T is a large subsemigroup of S, T and S have isomorphic maximal group homomorphism images; note that, since T is full, $\sigma_T = (T \times T) \cap \sigma_S$.

For example, if, with the notation of Proposition 1.3, $G' = G$ then T is a large subsemigroup of $P = P(H,\mathcal{X},\mathcal{Y})$. Indeed we have

Proposition 1.4. *Let S be an inverse semigroup with semilattice of idempotents E and maximum group homomorphic image G. If S can be embedded in a P-semigroup then S is isomorphic to a large subsemigroup of $P(G, \mathcal{X}, E)$ for some partially ordered set $\mathcal{X}$; in fact, $\mathcal{X}$ can be chosen so that $\mathcal{X} = G \cdot E$.*

Proof. Suppose $S \subseteq P(G, \mathcal{Z}, \mathcal{Y})$; then $\{A \in \mathcal{Y}\colon (A, 1) \in S\}$, being the set of idempotents of S, is a subsemilattice of $\mathcal{Y}$ isomorphic to E. Further, since $S \subseteq P(H, \mathcal{Z}, \mathcal{Y})$ and $E \subseteq \mathcal{Y}$ we clearly have $S \subseteq P(H, \mathcal{Z}, E)$ as a full subsemigroup.

Since $G \approx \{h \in H\colon (A, h) \in S$ for some $A \in E\}$, we can regard G as acting on $\mathcal{Z}$ and, from the form of the elements in $S \subseteq P(H, \mathcal{Z}, E)$ we see that, in fact, $S \subseteq P(G, \mathcal{Z}, E)$ as a large subsemigroup.

Finally, let $\mathcal{X} = G \cdot E \subseteq \mathcal{Z}$. Then, from the definitions of $P(G, \mathcal{Z}, E)$ and $P(G, \mathcal{X}, E)$, it is clear that $P(G, \mathcal{X}, E) = P(G, \mathcal{Z}, E)$. Hence $S \subseteq P(G, \mathcal{X}, E)$ with $\mathcal{X} = G \cdot E$.

Many important examples of P-semigroups arise by taking $\mathcal{X} = 2^X$ to be the set of subsets of a set X, G to be a subgroup of the symmetric group on X, acting on $\mathcal{X}$ by $g \cdot A = Ag^{-1}$ (G acts on the left, maps act on the right) and $\mathcal{Y}$ to be a subsemilattice of $\mathcal{X}$ under intersection. As we shall see in the next section, these P-semigroups are closely related to $\mathcal{I}_X$.

2. The division theorem. Munn's structure theory [11], [13] for inverse semigroups can be described as follows: every inverse semigroup S, with semilattice of idempotents E, has an idempotent separating homomorphic image which is a full subsemigroup of T_E. Diagramatically,

$$\begin{array}{ccc} & & T_E \\ & & \uparrow \text{full} \\ S & \xrightarrow{\text{i. s.}} & S/\mu \end{array}$$

where i.s. denotes idempotent separating.

The main theorem of this section, and the paper, shows that if S is an inverse semigroup, then there is a P-semigroup $P = P(G, \mathcal{X}, \mathcal{Y})$ such that S is an idempotent separating homomorphic image of a full (large) subsemigroup T of P. This theorem is dual to Munn's theorem mentioned above. This is seen by considering our theorem in diagramatic form.

$$\begin{array}{ccc} P & & \\ \text{full} \uparrow & & \\ T & \xrightarrow{\text{i. s.}} & S \end{array}$$

In order to prove the main theorem, we first investigate the semigroups

$P(G,\mathcal{X},\mathcal{Y})$ where $\mathcal{X} = 2^X$ and G is a subgroup of the symmetric group on X for some set X.

Lemma 2.1. *θ: $P(G,\mathcal{X},\mathcal{Y}) \to \mathcal{I}_X$ defined by $(A,g)\theta = g|A$ is an idempotent separating homomorphism of P into $\mathcal{I}_X$.*

Proof. Firstly, θ is clearly a map of $P = P(G,\mathcal{X},\mathcal{Y})$ into $\mathcal{I}_X$.
Let (A,g), $(B,h) \in P$. Then

$$\begin{aligned}(A,g)\theta(B,h)\theta &= (g|A)(h|B) = gh|(Ag \cap B)g^{-1}\\ &= gh|A \cap Bg^{-1} = gh|(A \cap gB)\\ &= (A,g)(B,h)\theta.\end{aligned}$$

Hence θ is a homomorphism.

Further, since $(A, 1)\theta$ is the identity on A, θ is clearly idempotent separating.

Lemma 2.2. *If G is the symmetric group on X then* $\operatorname{Im}\theta = \{\alpha \in \mathcal{I}_X\colon |X\backslash\Delta\alpha| = |X\backslash\nabla\alpha|, \Delta\alpha, \nabla\alpha \in \mathcal{Y}$ *and α induces an order isomorphism of* $\{B \in \mathcal{Y}\colon B \subseteq \Delta\alpha\}$ *onto* $\{C \in \mathcal{Y}\colon C \subseteq \nabla\alpha\}\}$.

Proof. Suppose $\alpha = (A,g)\theta$. Then $\alpha = g|A$ is the restriction of g to a map of A onto Ag and so $|X\backslash\Delta\alpha| = |X\backslash A| = |(X\backslash A)g| = |X\backslash Ag| = |X\backslash\nabla\alpha|$. Further, from the definition of P, $(A,g) \in P$ implies A, $g^{-1}A = Ag \in \mathcal{Y}$. Suppose $B \in \mathcal{Y}$, $B \subseteq A$. Then $B\alpha = Bg = g^{-1}B = g^{-1}(B \cap A) \in \mathcal{Y}$ and $B\alpha \subseteq \nabla\alpha$. Hence the map $B \mapsto B$ is a one-to-one order preserving map of $\{B \in \mathcal{Y}\colon B \subseteq \Delta\alpha\}$ into $\{C \in \mathcal{Y}\colon C \subseteq \nabla\alpha\}$. Similarly, the map $C \mapsto C\alpha^{-1}$ is a one-to-one order preserving map of $\{C \in \mathcal{Y}\colon C \subseteq \nabla\alpha\}$ into $\{B \in \mathcal{Y}\colon B \subseteq \Delta\alpha\}$. Since these maps are inverses of one another, it follows that α induces an order isomorphism of $\{B \in \mathcal{Y}\colon B \subseteq \Delta\alpha\}$ onto $\{C \in \mathcal{Y}\colon C \subseteq \nabla\alpha\}$.

Conversely, suppose that α is in the right side of the equation for $\operatorname{Im}\theta$ and let $A = \Delta\alpha$. Since $|X\backslash\Delta\alpha| = |X\backslash\nabla\alpha|$ there exists $g \in G$ with $\alpha = g|A$. We claim $(A,g) \in P$. Then, since $\alpha = g|A$, $\alpha \in \operatorname{Im}\theta$. Let $B \in \mathcal{Y}$. Then

$$A \cap gB = (Ag \cap B)g^{-1} = (A\alpha \cap B)\alpha^{-1} \in \mathcal{Y}$$

since $A\alpha \cap B \in \mathcal{Y}$ ($\mathcal{Y}$ is a subsemilattice of $\mathcal{X}$) and α^{-1} induces an isomorphism of $\{C \in \mathcal{Y}\colon C \subseteq A\alpha\}$ onto $\{B \in \mathcal{Y}\colon B \subseteq A\}$. Likewise $g^{-1}(A \cap B) = (A \cap B)\alpha \in \mathcal{Y}$. Hence $(A,g) \in P$.

Because of the relationship between $P(G, \mathcal{X}, \mathcal{Y})$ and $\mathcal{I}_X$, every representation of an inverse semigroup S is connected with a P-semigroup. This relationship is made precise for faithful representations in the next proposition.

Proposition 2.3. *Let ρ: $S \to \mathcal{I}_Z$ be a faithful representation of an inverse semigroup S by one-to-one partial transformations of a set Z. Let $X = Y \cup Z$ where $Y = Z$ if Z is finite and otherwise $Y \cap Z = \square$, $|Y| = |Z|$. Then S is an idempotent separating homomorphic image of a full subsemigroup of $P(G, \mathcal{X}, \mathcal{Y})$ where G is the symmetric group on X, $\mathcal{X} = 2^X$ and $\mathcal{Y} = \{A \subseteq X\colon A = \Delta\rho_a$ for some $a \in S\}$.*

Proof. We regard ρ as a representation of S by one-to-one partial transformations of X; this can clearly be done since $Z \subseteq X$.

Let $\alpha = \rho_a$; then $|X\backslash\Delta\alpha| = |Y| = |X| = |X\backslash\nabla\alpha|$ if Z is infinite, while $|X\backslash\Delta\alpha| = |Z\backslash\Delta\alpha| = |Z\backslash\nabla\alpha| = |X\backslash\nabla\alpha|$ if Z is finite since α is a one-to-one map of $\Delta\alpha$ onto $\nabla\alpha$. Further, if $B \in \mathcal{Y}$, $B \subseteq \Delta\alpha$, then $B = \Delta\rho_e$ for some idempotent e and $B\alpha = B\rho_{ea} = \nabla\rho_{ea} \in \mathcal{Y}$ so that α induces a one-to-one order preserving map of $\{B \in \mathcal{Y}\colon B \subseteq \Delta\alpha\}$ into $\{C \in \mathcal{Y}\colon C \subseteq \nabla\alpha\}$. Similarly, since $\nabla\alpha = \Delta\alpha^{-1}$, α^{-1} induces a one-to-one order preserving map of $\{C \in \mathcal{Y}\colon C \subseteq \nabla\alpha\}$ into $\{B \in \mathcal{Y}\colon B \subseteq \Delta\alpha\}$. Since these maps are inverses of one another, we conclude that $\alpha \in \operatorname{Im}\theta$ where $\theta\colon P(G,\mathcal{X},\mathcal{Y}) \to \mathcal{I}_X$ is defined by $(A,g)\theta = g|A$. Hence $S\rho \subseteq \operatorname{Im}\theta$.

Let $T = S\rho\theta^{-1}$; then T is an inverse subsemigroup of $P = P(G,\mathcal{X},\mathcal{Y})$ and $\theta\rho^{-1}$ is an idempotent separating homomorphism of T onto S. Let $A \in \mathcal{Y}$; then $A = \Delta\rho_e$ for some $e^2 = e \in S$. Hence $(A,1)\theta = \rho_e \in S\rho$ and so $(A,1) \in T$. Hence T contains all the idempotents of P and so is a full subsemigroup T of P.

The Preston-Vagner representation [16], [21] is a faithful representation for each inverse semigroup S. Hence, as an immediate consequence of Proposition 2.3, we have

Theorem 2.4. *Let S be an inverse semigroup. Then S is an idempotent separating homomorphic image of a full subsemigroup T of $P(G, \mathcal{X}, \mathcal{Y})$ for some G, $\mathcal{X}$, $\mathcal{Y}$.*

Note. Since T is a full subsemigroup of $P(G, \mathcal{X}, \mathcal{Y})$ and the homomorphism $T \to S$ is idempotent separating, S and P have isomorphic semilattices of idempotents. Thus $\mathcal{Y} \approx E$, the semilattice of idempotents of S.

An inverse semigroup S is called *proper* if $ae = e = e^2$ implies $a = a^2$ for each $a \in S$. By the proof of Proposition 2.3, each $P(G, \mathcal{X}, \mathcal{Y})$ is proper and so are its inverse subsemigroups. Hence

Corollary 2.5. *Every inverse semigroup is an idempotent separating homomorphic image of a proper inverse semigroup.*

Corollary 2.5 shows, in particular, that the free inverse semigroup $I(X)$, on a set X, is proper. Further, it is easy to see from the freeness of $I(X)$ and Proposition 2.3, that $I(X)$ is a full subsemigroup of $P(G, \mathcal{X}, \mathcal{Y})$ for some $\mathcal{X}$, $\mathcal{Y}$ where G is the free group on X. By analysing the embedding, one can obtain the structure theorem for free inverse semigroups due to Scheiblich [20]; cf. [9].

3. **Semilattices of groups.** Let $S = \bigcup\{G_e\colon e \in E\}$ be a semilattice of groups with semilattice E of idempotents. According to Theorem 2.4, S is an idempotent separating homomorphic image of a full subsemigroup T of $P(G, \mathcal{X}, E)$ for some G, $\mathcal{X}$. Since S is a semilattice of groups, so is T. Hence, for $(A,g) \in T$, $(A,g)(g^{-1}A,g^{-1}) = (g^{-1}A,g^{-1})(A,g)$; that is $(g^{-1}A,1) = (A,1)$. Thus $(A,g) \in T$ implies $A = gA$.

For any $B \in E$, we have $(B,1) \in T$ and so $(A,g)(B,1) \in T$. Hence $(A \wedge gB, g) \in T$ which implies $g^{-1}(A \wedge gB) = A \wedge gB$; that is, since $g^{-1}A = A$,

$A \wedge B = A \wedge gB$ for all $B \in E$. It follows that, for $(A,g)(B,h) \in T$,

$$(A,g)(B,h) = (A \wedge B, gh).$$

Hence T is isomorphic to a full subsemigroup of $E \times G$. This shows that S is isomorphic to an idempotent separating homomorphic image of a full subsemigroup of $E \times G$ for some group G.

In this section, we give a direct proof of this result by constructing such a division for $S = \bigcup\{G_e : e \in E\}$. First we characterise subsemigroups of $E \times G$.

Proposition 3.1. *Let* $S = \bigcup\{G_e : e \in E\}$ *be a semilattice of groups with linking homomorphisms* $\theta_{e,f}$, $e \geq f$, *and let* $G = \varinjlim G_e$ *be the direct limit of the directed set* $\theta_{e,f}\colon G_e \to G_f$. *Then the following are equivalent*:

(i) *S is proper.*

(ii) *Each $\theta_{e,f}$ is one-to-one .*

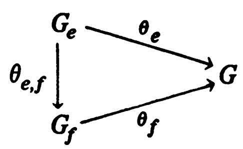
(iii) *S is isomorphic to a large subsemigroup of $E \times G$.*

(iv) *S is isomorphic to a subsemigroup of $P(H, \mathcal{X}, E)$ for some group H and partially ordered set $\mathcal{X}$.*

Proof. Clearly (iii) ⇒ (iv) ⇒ (i) so we need only show (i) ⇒ (ii) ⇒ (iii).

(i) ⇒ (ii). Let $a \in G_e$, $f \leq e$, and suppose $a \in \operatorname{Ker} \theta_{e,f}$. Then $af = a\theta_{e,f} = f$. Hence, since S is proper, a is idempotent; thus $a = e$. This shows that $\theta_{e,f}$ is one-to-one.

(ii) ⇒ (iii). Let $\theta_e\colon G_e \to G$ be the natural homomorphisms $G_e \to G = \varinjlim G_e$ induced by the $\theta_{e,f}$; thus each diagram



$e \geq f$, commutes. Since each $\theta_{e,f}$ is one-to-one, so is each θ_e.

Define $\theta\colon S \to E \times G$ by

$$s\theta = (e, s\theta_e) \quad \text{if } s \in G_e.$$

Then θ is well defined and one-to-one. Further, if $s \in G_e$, $t \in G_f$,

$$\begin{aligned} s\theta t\theta &= (e, s\theta_e)(f, t\theta_f) = (ef, s\theta_e t\theta_f) \\ &= (ef, s\theta_{e,ef}\theta_{ef} t\theta_{f,ef}\theta_{ef}) \\ &= (ef, (s * t)\theta_{ef}) = (s * t)\theta \end{aligned}$$

where $s * t$ denotes the product of s and t in S. Hence θ is an isomorphism of S onto a subsemigroup T of $E \times G$. Finally, for $e \in E$, $(e, 1) = (e, e\theta_e) = e\theta$, so T is full and, for each $g \in G$, $g = (e, g_e)\theta$ for some $e \in E$, $g_e \in G_e$, so that, in fact, T is large.

Suppose now $S = \bigcup\{G_e: e \in E\}$ is not necessarily proper. It follows, from Proposition 3.1, that in order to obtain a division

$$\begin{array}{ccc} E \times G & & \\ \text{full} \uparrow & & \\ T & \xrightarrow{\text{i. s.}} & S \end{array}$$

we need only find a semilattice of groups $T = \bigcup\{K_e: e \geq f\}$ with one-to-one linking homomorphisms $\phi_{e,f}$, $e \geq f$, and an idempotent separating homomorphism ψ from T onto S.

Lemma 3.2. *For each $e \in E$, let $K_e = \Pi\{G_f: f \not\leq e\}$ and, for each e, f with $e \geq f$ define $\phi_{e,f}$: $K_e \to K_f$ by $(g_\gamma)\phi_{e,f} = (h_\delta)$ where*

$$\begin{aligned} h_\delta &= g_\delta, & \delta &\not< e, \\ &= g_e\theta_{e,\delta}, & \delta &\leq e. \end{aligned}$$

Then each $\phi_{e,f}$ is a one-to-one homomorphism and $\{\phi_{e,f}: e \geq f\}$ is a directed set of homomorphisms.

Proof. That each $\phi_{e,f}$ is a one-to-one homomorphism is clear. Suppose $e \geq f \geq g$ and let $(u_\gamma) \in K_e$. Then

$$\begin{aligned} (u_\gamma)\phi_{e,f}\phi_{f,g} &= (v_\delta)\phi_{f,g} \quad \text{where } v_\delta &&= u_\delta, && \delta \not< e, \\ & && = u_e\theta_{e,\delta}, && \delta \leq e; \\[1em] &= (k_\varepsilon) \quad \text{where } k_\varepsilon && = v_\varepsilon, && \varepsilon \not< f, \\ & && = v_f\theta_{f,\varepsilon}, && \varepsilon \leq f. \end{aligned}$$

Then

$$\begin{aligned} k_\varepsilon &= u_\varepsilon, & & \varepsilon \not< e, \\ &= u_e\theta_{e,\varepsilon}, & & \varepsilon \leq e, \varepsilon \not< f, \\ &= u_e\theta_{e,f}\theta_{f,\varepsilon}, & & \varepsilon \leq f, \\[1em] &= u_\varepsilon, & & \varepsilon \not< e, \\ &= u_e\theta_{e,\varepsilon}, & & \varepsilon \leq e. \end{aligned}$$

Hence $(u_\gamma)\phi_{e,f}\phi_{f,g} = (u_\gamma)\phi_{e,g}$; that is $\phi_{e,f}\phi_{f,g} = \phi_{e,g}$. Finally, for each $e \in E$, ϕ_{ee} is the identity on K_e.

Lemma 3.3. *Define $\psi: T \to S$, where $T = \bigcup\{K_e: e \in E\}$ is semilattice of groups with linking homomorphisms $\phi_{e,f}$, by*

$$t\psi = t\pi_e, \quad \textit{if } t \in K_e,$$

where π_e is the projection $K_e \to G_e$. Then ψ is an idempotent separating homomorphism of T onto S.

Proof. ψ is clearly an idempotent separating map of T onto S. Let $u, v \in T$ with $u = (u_\gamma) \in T_e$, $v = (u_\gamma) \in T_f$; then

$$\begin{aligned}(u * v)\psi &= (u\phi_{e,ef}v\phi_{f,ef})\psi = (u\phi_{e,ef}v\phi_{f,ef})\pi_{ef} \\ &= u\phi_{e,ef}\pi_{ef}v\phi_{f,ef}\pi_{ef} \\ &= u_e\theta_{e,ef}v_f\theta_{f,ef} = u\psi * v\psi\end{aligned}$$

where $*$ denotes the product in S, T. Hence ψ is a homomorphism.

Remark. Suppose that $S = \bigcup\{G_e: e \in E\}$ is a finite chain of groups and let ρ be the regular representation of S. Then, by Proposition 2.3, S is an idempotent separating homomorphic image of a full subsemigroup of $P(G, \mathcal{X}, \mathcal{Y})$ where $\mathcal{X} = 2^S$, $\mathcal{Y} = \{Sa: a \in S\}$ and G is any subgroup of the symmetric group on S such that, for each $a \in S$, there exists $\alpha \in G$ with $\alpha \mid Sa = \rho_a$.

For each $a \in S$ define $\rho'_a: S \to S$ by

$$\begin{aligned}x\rho'_a &= x, \qquad x \notin Sa, \\ &= xa, \qquad x \in Sa.\end{aligned}$$

Then ρ'_a is a permutation of S which extends ρ_a so we may take G to be the group generated by the ρ'_a, $a \in S$.

For each $e \in E$, the group R_e of all $\rho = \eta_g$ such that

$$\begin{aligned}x\rho &= x, \qquad x \notin G_e, \\ &= x\rho_g, \qquad x \in G_e, \quad \text{for some } g \in G_e,\end{aligned}$$

is isomorphic to G_e. Further, each ρ'_a belongs to the subgroup $\prod R_e$ of the symmetric group on S; for if $a \in S$, $\rho'_a = \prod\{\eta_{ae}: e \leq aa^{-1}\}$. On the other hand, if $a \in S$ and $aa^{-1} = e$ and e covers f in E then

$$\begin{aligned}x\rho_a\rho'_{fa^{-1}} &= x, \qquad x \notin Sa, \\ &= xa, \qquad x \in Sa\backslash Sf, \\ &= x, \qquad x \in Sf; \\ \\ &= xa, \qquad x \in G_e, \\ &= x, \qquad \text{otherwise}; \\ \\ &= x\eta_a.\end{aligned}$$

Hence $R_e \subseteq G$ for each $e \in E$ and so $G = \prod R_e \approx \prod G_e$.

Since E is finite, $\prod G_e = \varinjlim K_e$ where K_e is as in Lemma 3.3. Hence the division obtained using the regular representation in this natural way is the same as that contructed explicitly in this section.

4. Bisimple inverse semigroups. In this section we provide an explicit division for bisimple inverse semigroups. In fact, we show that proper bisimple inverse semigroups are P-semigroups and we obtain a structure theorem for these semigroups. This theorem generalises the structure theorem for bisimple inverse monoids which was given in [9]. We shall use Reilly's theory [18] of bisimple inverse semigroups in terms of RP-systems with which we assume familiarity in content, terminology and notation.

Lemma 4.1. *Let S be a bisimple inverse semigroup, let R be an $\mathcal{R}$-class of S with idempotent e and let $P = R \cap eSe$. If S is proper and $\sigma^\natural$ is the canonical homomorphism of S onto its maximal group homomorphic image G then $\sigma^\natural \mid R$ is one-to-one.*

Proof. Suppose $a, b \in R$ and $a\sigma^\natural = b\sigma^\natural$. Then $aa^{-1} = bb^{-1}$ and $(a,b) \in \sigma$. Hence by [9], $a = b$.

It follows from Lemma 4.1 that we can assume $R \subseteq G$. Let $\mathcal{X} = \{Pg\colon g \in G\}$, $\mathcal{Y} = \{Pa\colon a \in R\}$. Then $\mathcal{X}$ is a partially ordered set under inclusion having $\mathcal{Y}$ as an ideal and subsemilattice. Further, G acts transitively on $\mathcal{X}$ by $h \cdot Pg = Pgh^{-1}$. We shall prove that $S \approx P(G,\mathcal{X},\mathcal{Y})$.

Each element of S is of the form $a^{-1}b$ with $a, b \in R$ and, by [18],

$$a^{-1}b = c^{-1}d \text{ in } S \Leftrightarrow a^{-1}a = a^{-1}b(a^{-1}b)^{-1} = c^{-1}d(c^{-1}d)^{-1} = c^{-1}c$$
$$\text{and } a^{-1}b = c^{-1}d \text{ in } G$$
$$\Leftrightarrow Pa = Pc \text{ and } a^{-1}b = c^{-1}d \text{ in } G.$$

Hence $\theta\colon S \to P(G,\mathcal{X},\mathcal{Y})$ given by $(a^{-1}b)\theta = (Pa, a^{-1}b)$ is a well-defined one-to-one map.

Lemma 4.2. *θ is an isomorphism of S onto $P(G, \mathcal{X}, \mathcal{Y})$.*

Proof. Suppose $a^{-1}b, c^{-1}d \in S$; then

$$\begin{aligned}(a^{-1}b)\theta(c^{-1}d)\theta &= (Pa, a^{-1}b)(Pc, c^{-1}d)\\ &= (Pa \cap Pcb^{-1}a, a^{-1}bc^{-1}d)\\ &= ((Pb \cap Pc)b^{-1}a, [(c * b)a]^{-1}(b * c)d)\\ &= (P(c * b)a, [(c * b)a]^{-1}(b * c)d)\\ &= (a^{-1}b)(c^{-1}d)\theta\end{aligned}$$

where $c * b$ and $b * c$ are as in [18]. Thus θ is a homomorphism.

We have already pointed out that θ is one-to-one so that it only remains to show that it is onto. Since $P(G, \mathcal{X}, \mathcal{Y})$ is an ideal P-semigroup, $(Pa, g) \in P(G, \mathcal{X}, \mathcal{Y})$ if and only if Pa, $g^{-1} \cdot Pa = Pag$ both belong to $\mathcal{Y}$; that is, if and only if $a, ag \in R$. Hence, if $b = ag$, $a^{-1}b \in S$ and $(a^{-1}b)\theta = (Pa, g)$. Thus θ is onto.

Theorem 4.3. *Let E be a uniform semilattice and let G be a group. Then a semigroup S is a proper bisimple inverse semigroup with semilattice E and maximal group homomorphic image G if and only if $S \approx P(G, \mathcal{X}, E)$ for some (uniform) partially ordered set $\mathcal{X}$, having E as an ideal, and transitive action of G on $\mathcal{X}$ such that for each $g \in G$ there exists $A \in E$ with $g^{-1}A \in E$.*

Proof. Suppose that $S \approx P(G, \mathcal{X}, E)$; then, because E is an ideal and G acts transitively, it follows from Proposition 1.2 and Proposition 1.3 that S is a proper bisimple inverse semigroup with semilattice E and maximal group homomorphic image G.

The converse is immediate from Lemma 4.1 and Lemma 4.2.

Since the semigroup $P(G, E, E)$, where G acts transitively on E, are particularly simple examples of P-semigroups, it is of interest to characterise them.

Corollary 4.4. *Let E be a semilattice and let G be a group which acts transitively on E by (order) automorphisms. Then $S = P(G, E, E)$ is a bisimple inverse semigroup. Further, if R is an $\mathcal{R}$-class of S, $\sigma^{\natural}$ induces a one-to-one map of R onto G.*

Conversely, suppose that S is a bisimple inverse semigroup with semilattice of idempotents E and let R be an $\mathcal{R}$-class of S. If $\sigma^{\natural}$ induces a one-to-one map of R onto G then $S \approx P(G, E, E)$ for some transitive action of G on E.

Proof. That S is a bisimple inverse semigroup is immediate from Proposition 1.2. Since S has maximal group homomorphic image G and is proper, $\sigma^{\natural}$ induces a one-to-one map of R into G by Lemma 4.1. Suppose $g \in G$, and $(A, 1)$ is the idempotent in R; then $(A, g) \in S$. Further, since $(A, g)(A, g)^{-1} = (A, 1)$, $(A, g) \in R$. But $(A, g)\sigma^{\natural} = g$. Hence $\sigma^{\natural} \mid R$ is onto G.

Conversely, suppose that $\sigma^{\natural} \mid R$ is one-to-one onto. Then, firstly, S is proper. For, suppose $c^{-1}ca^{-1}b = c^{-1}c$, $a, b, c \in R$; then $c = ca^{-1}b$ and so, multiplying by $a * c$,

$$(c * a)a = a \wedge c = (a * c)c = (a * c)a^{-1}b = (c * a)b.$$

Thus $a\sigma^{\natural} = b\sigma^{\natural}$ so that, since $a, b \in R$, $a = b$. Hence $a^{-1}b$ is idempotent.

It follows, from Theorem 4.3, that $S \approx P(G, \mathcal{X}, E)$ where $\mathcal{X} = \{Pg\colon g \in G\}$ and $E = \{Pg\colon g \in R\}$. But, since $\sigma^{\natural} \mid R$ is onto, $\mathcal{X} = E$. Hence $S \approx P(G, E, E)$.

Reilly [17] has shown that any bisimple ω-semigroup can be described as follows. Let H be a group and α any endomorphism of H and set $S = \mathbf{Z}^+ \times H \times \mathbf{Z}^+$ under the multiplication

$$(m,g,n)(r,h,s) = (r \vee n - n + m, g\alpha^{r \vee n - n} h\alpha^{n \vee r - r}, n \vee r - r + s).$$

Then S is a bisimple ω-semigroup with group of units isomorphic to H and any such semigroup has this form; S is denoted by $B(H, \alpha)$.

It is easy to see that $B(H, \alpha)$ is proper if and only if α is one-to-one. Further, Munn and Reilly [15] have shown that an idempotent separating homomorphism θ of $B(K, \beta)$ onto $B(H, \alpha)$ is given by

$$(m,g,n)\theta = (m, g\phi, n)$$

where ϕ is a homomorphism of K onto H such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\ \beta\ } & K \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ H & \xrightarrow{\ \alpha\ } & H \end{array}$$

commutes. It follows from Theorem 4.3, that, if we wish to construct explicitly a division, as in Theorem 2.4, for $S = B(H, \alpha)$, we need only construct K, β, ϕ such that $\phi\alpha = \beta\phi$ with ϕ onto and β one-to-one.

Lemma 4.5. *Let H be a group and let α be an endomorphism of H. Define $K = H \times H \times \cdots$ and β: $K \to K$ by*

$$(h_1, h_2, \ldots)\beta = (h_1\alpha, h_1, h_2, \ldots).$$

Then β is a one-to-one homomorphism and, if ϕ: $K \to H$ is defined by

$$(h_1, h_2, \ldots)\phi = h_1,$$

ϕ is a homomorphism of K onto such that $\phi\alpha = \beta\phi$.

Proof. This is straightforward.

Note. Even if H is finite, K, as constructed in Lemma 4.5, is infinite. This is necessary unless $S = B(H, \alpha)$ is proper. For suppose that K is finite and that

$$\begin{array}{ccc} K & \xrightarrow{\ \beta\ } & K \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ H & \xrightarrow{\ \alpha\ } & H \end{array}$$

commutes where β is one-to-one and ϕ is onto. Then, since K is finite, β is also onto and thus so is $\phi\alpha = \beta\phi$. Hence α is onto and so, because $H = K\phi$ must be finite, α is one-to-one.

If S is a proper bisimple ω-semigroup then, by Theorem 4.3, $S \approx P(G, \mathcal{X}, E)$ where E is an ω-chain, $\mathcal{X}$ is a partially ordered set having E as an ideal and G acts transitively on $\mathcal{X}$ in such a way that $\mathcal{X}$ is the set of translates of E under G. It follows that $\mathcal{X}$ is an ω-tree; that is each principal ideal of $\mathcal{X}$ is an ω-chain. On the other hand if $\mathcal{X}$ is an ω-tree and E is a principal ideal of $\mathcal{X}$ then $P(G, \mathcal{X}, E)$ is a bisimple ω-semigroup. Hence we have

Proposition 4.6. *Let $\mathcal{X}$ be an ω-tree and let G be a group which acts transitively on $\mathcal{X}$ by order automorphisms; if E is a principal ideal of $\mathcal{X}$ then $P(G, \mathcal{X}, E)$ is a proper bisimple ω-semigroup. Each proper bisimple ω-semigroup is of this form.*

Remark. If E is required only to be an ideal of $\mathcal{X}$, $P(G, \mathcal{X}, E)$ is a proper bisimple inverse semigroup on the ω-tree E; each such also has this form.

ω-trees which have a transitive automorphism group are rather special; for example it is easy to see that they can have no maximal elements. The simplest ω-tree which has a transitive automorphism group is $\mathbf{Z}$; its automorphism group is isomorphic to $\mathbf{Z}$. To end this section, we characterise the bisimple semigroups $P(G, \mathbf{Z}, \mathbf{Z}^-)$.

Theorem 4.7. *Let $S = B(H, \theta)$ be a bisimple ω-semigroup. Then $S \approx P(G, \mathbf{Z}, \mathbf{Z}^-)$ for some transitive group G of automorphisms of $\mathbf{Z}$ if and only if θ is an automorphism of H.*

Proof. Suppose θ is an isomorphism, and let $a = (0, e, 1)$ where e is the identity of H and identify H with the set of triples $(0, g, 0)$. Then [15, Corollary 3.6] G is the semidirect product of H by $\mathbf{Z}$ with linking automorphism θ; thus the elements of G can be represented as ga^n, $g \in H$, $n \in \mathbf{Z}$, under $ga^n ha^m = g(h\theta^n)a^{n+m}$. Under this representation, P (see Lemmas 4.1, 4.2) corresponds to $\{ga^n : n \geq 0\}$ and $Pga^n = \{ha^{m+n} : m \geq 0\}$. Hence the set $\mathcal{X}$ of translates of P in G is a chain and thus is isomorphic to $\mathbf{Z}$. Thus, by Lemma 4.2, $S \approx P(G, \mathbf{Z}, \mathbf{Z}^-)$.

Conversely suppose $S = P(G, \mathbf{Z}, \mathbf{Z}^-)$ and let $H = \{g \in G : g \cdot n = n$ for all $n \in \mathbf{Z}\}$. Then $H \Delta G$ and, since Aut $\mathbf{Z} \approx \mathbf{Z}$, $G/H \approx \mathbf{Z}$. Hence G is a semidirect product of H by G; thus we may suppose $G = H \times \mathbf{Z}$ under

$$(g, m)(h, n) = (g(h\theta^m), m + n)$$

where θ is an automorphism and G acts on $\mathbf{Z}$ by $(g, m) \cdot n = m + n$. Let $(g, m) \in G$; then we have $(g, m)^{-1} = (g^{-1}\theta^{-m}, -m)$. Hence

$$S = P = \{(n, g, m) : n \leq 0, m\}.$$

Further S has group of units $\{(g, 0) : g \in H\} \approx H$. Let $a = (0, (e, 1))$; then $aa^{-1} = (0, e, 0)$, $a^{-1}a = (-1, e, 0)$ so that $a^{-1}a$ is covered by the identity. Hence, from [17], $S \approx B(H, \phi)$ where $\phi: H \to H$ is defined by

$$\begin{aligned}(0, g\phi, 0) &= a(0, g, 0)a^{-1} \\ &= (0, e, 1)(0, g, 0)(-1, e, -1) \\ &= (0, g\theta, 1)(-1, e, -1) \\ &= (0, g\theta, 0),\end{aligned}$$

that is $S \approx B(H, \theta)$ where θ is an isomorphism.

Corollary 4.8. *If S is a proper bisimple inverse semigroup with a finite group of units H then $S \approx P(G, \mathbf{Z}, \mathbf{Z}^-)$; where G is an extension of H by $\mathbf{Z}$. In particular, the bicyclic semigroup is isomorphic to $P(\mathbf{Z}, \mathbf{Z}, \mathbf{Z}^-)$.*

Finally, we can use Theorem 4.7 to characterise those bisimple ω-semigroups $B(H, \theta)$ with θ onto.

Theorem 4.9. *Let $S = B(H, \theta)$ be a bisimple ω-semigroup. Then S is an idempotent separating homomorphic image of $P(G, \mathbf{Z}, \mathbf{Z}^-)$ for some transitive group G of automorphisms of $\mathbf{Z}$ if and only if θ is onto.*

Proof. If S is an idempotent separating homomorphic image of $P(G, \mathbf{Z}, \mathbf{Z}^-) \approx B(K, \phi)$ then ϕ is an isomorphism, by Theorem 4.7, and, by [15], there is a homomorphism η of K onto H such that $\phi\eta = \eta\theta$. Hence θ is onto.

Conversely, if θ is onto, let K be the subgroup of $H \times H \times \cdots$ consisting of all $(g_1, g_2, \ldots)$ such that $g_i = g_{i+1}\theta$, $i = 1, 2, \ldots$. Define ϕ: $K \to K$ and η: $K \to H$ by

$$(g_1, g_2, \ldots)\phi = (g_1\theta, g_1, g_2, \ldots), \qquad (g_1, g_2, \ldots)\eta = g_1,$$

respectively. Then ϕ is an automorphism of K and η is a homomorphism of K onto H such that $\phi\eta = \eta\theta$. Hence $B(H, \theta)$ is an idempotent separating homomorphic image of $B(K, \phi)$. But, by Theorem 4.7, $B(K, \phi) \approx P(G, \mathbf{Z}, \mathbf{Z}^-)$ for some group G acting transitively on $\mathbf{Z}$.

5. **Miscellaneous results and concluding remarks.**

5.1. *Brandt semigroups.* In §§3, 4, we have given explicit divisions for semilattices of groups and ω-bisimple inverse semigroups. Here we give one for Brandt semigroups.

Let $S = \mathscr{M}^\circ(G; I, I, \Delta)$ be a Brandt semigroup and denote by S_I the symmetric group on I. Then $K = G \times S_I$ acts on $I^\circ = I \cup \{0\}$ by $(g, \alpha) \cdot i = i\alpha^{-1}$, $(g, \alpha) \cdot 0 = 0$; this action is clearly by order automorphisms.

Define ϕ: $P(K, I^\circ, I^\circ) \to S$ by

$$(i, (g, \alpha))\phi = (i, g, i\alpha), \qquad (0, (g, \alpha))\phi = 0.$$

Then ϕ is clearly an idempotent separating mapping onto S. Further,

$$
\begin{aligned}
(i,(g,\alpha))\phi(j,(h,\beta))\phi &= (i,g,i\alpha)(j,h,j\beta) \\
&= (i,gh,j\beta) \qquad \text{if } i\alpha = j, \\
&= 0 \qquad\qquad\quad \text{otherwise,}
\end{aligned}
$$

while

$$
\begin{aligned}
[(i,(g,\alpha))(j,(h,\beta))]\phi &= (i \wedge j\alpha^{-1},(gh,\alpha\beta))\phi \\
&= (i,gh,j\beta) \qquad \text{if } i = j\alpha^{-1}, \\
&= 0 \qquad\qquad\quad \text{otherwise.}
\end{aligned}
$$

Hence $(i,(g,\alpha))\phi(j,(h,\beta))\phi = [(i,(g,\alpha))(j,(h,\beta))]\phi$. Since $\{(0,(g,\alpha))\colon g \in G, \alpha \in S_I\}$ is an ideal of $P(K,I^\circ,I^\circ)$ which is mapped by ϕ to the zero of S, ϕ also preserves the other products in $P(K,I^\circ,I^\circ)$. Hence ϕ is a homomorphism and S is in fact an idempotent separating homomorphic image of $P(K, I^\circ, I^\circ)$.

In §§3 and 4, it was shown that proper bisimple inverse semigroups and proper semilattices of groups can be embedded in P-semigroups. It can be shown that the same is true for finite proper inverse semigroups with two $\mathcal{J}$-classes; is this true in general? More precisely,

5.2. *Is every proper inverse semigroup a full subsemigroup of some P-semigroup*? The theory in §§3 and 4 shows that, in the cases considered there, P could be taken to be an ideal P-semigroup; this is also the case for finite inverse semigroups with two $\mathcal{J}$-classes. Further many important inverse semigroups are ideal P-semigroups; see [9]. Hence one may ask

5.3. *Is every inverse semigroup an idempotent separating homomorphic image of a full subsemigroup of an ideal P-semigroup*? Let E be a semilattice. Then, by Theorem 2.4, T_E the semigroup of isomorphisms between principal ideals of E divides $P = P(G,\mathcal{X},E)$ for some $G, \mathcal{X}$, such that each $x \in \mathcal{X}$ is of the form $g \cdot e$ for some $e \in E$. If E is an ideal of $\mathcal{X}$ (as occurs when E is uniform by Theorem 4.3) then any isomorphism between principal ideals of $\mathcal{X}$ is extendible to an automorphism of $\mathcal{X}$. For if α: $x\mathcal{X} \to y\mathcal{X}$ is such an isomorphism $\alpha = g\beta h$ where $x = g^{-1} \cdot e, y = h \cdot f, e, f \in E$ and β is an isomorphism $eE = e\mathcal{X} \to f\mathcal{X} = fE$. Since T_E divides P, there exists $\gamma \in G$ such that $\gamma \mid eE = \beta$. Thus α is the restriction of $g\gamma h$.

5.4. *What can be said about partially ordered sets with the property that each isomorphism between principal ideals can be extended to an automorphism*? Reilly's structure theorem for bisimple ω-semigroups can be regarded as giving an answer to 5.4 when the partially ordered set $\mathcal{X}$ is an ω-tree. For, let E be a principal ideal of $\mathcal{X}$ (and thus an ω-chain) and let G be the group of order automorphisms of $\mathcal{X}$. Then $P = P(G,\mathcal{X},E)$ is a bisimple inverse semigroup. Further, if $G' = \{g$

$\in G$: $gE \cap E \neq \square\} = \{g \in G$: $(A,g) \in P$ for some $A \in E\}$ and $\mathcal{X}' = G' \cdot E$, then $P = P(G', \mathcal{X}', E)$ and P has maximum group homomorphic image G'. Thus, by the proof of Theorem 4.3, $\mathcal{X}'$ is isomorphic to the set of translates in G' of the right unit subsemigroup of P.

It follows that one can use Reilly's theory to describe ω-trees $\mathcal{X}$ with the following properties for some principal ideal E.

(i) Every isomorphism between principal ideals of $\mathcal{X}$ can be extended to an automorphism of $\mathcal{X}$.

(ii) If $x \in \mathcal{X}$, there is an automorphism g of $\mathcal{X}$ such that $xg \in E$ and $Eg \cap E \neq \square$.

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